4.0 PREPARATION OF VISUAL OBSERVATIONS

In this chapter we develop the theory required to compare the observations of the satellites of Saturn with any dynamical theory. The task falls into three main parts :

1. Conversion of the time scales of the data into Universal Time and then into Ephemeris Time, which is the time scale in which the dynamical theories are expressed. We also consider one or two other minor points relating to time- scales.

2. The calculations required to obtain a computed topocentric position of a satellite from its Saturnicentric theory. Such a topocentric position must be directly comparable to observations and so it must incorporate a number of important physical effects in addition to the simple translation of the origin from Saturn to the observer.

3. In order to enter a differential correction process on the parameters of the theory it is necessary to calculate the partial derivatives of the observed quantities with respect to the Saturnicentric coordinates of the satellite.

The solutions to these problems are covered in the following sections.

4.1 SYSTEMS OF TIME MEASUREMENT

Several time scales are used in the data sets. In each case we must devise a procedure for converting the given time of each observation into Universal Time. The only case where any complication might arise is in the treatment of Local Apparent Sidereal Time, although some care is also required when dealing with Mean Astronomical Time systems (such as WMAT). The method of conversion from LST to UT and from WMAT to UT is given in the next two sub-sections. As will be shown in a subsequent section, we require the Local Sidereal Time of each observation regardless of the time scale used for the observation itself and so a method for calculating LST from UT is also given.

The dynamical theories of the satellites of Saturn have as their independent variable yet another time scale, known as Ephemeris Time (E.T.). The significance of this concept is rather important and will be discussed in a further sub-section where we also consider the effect of errors in the time argument as they are reflected in the observed positions of the satellites.

The final sub-section deals with the correction to the 'observed' time to allow for the 'light-time' delay. The practical method for making this correction involves an iteration to determine the topocentric position of Saturn and is thus also relevant to the section on reference frames.

4.1.1 CONVERSION OF WMAT TO UNIVERSAL TIME

Washington Mean Astronomical Time is a solar time scale which runs exactly 12 hours behind Washington Mean Civil Time. An astronomical day begins at midday on the corresponding civil day : thus the astronomical day 1875 February 7 begins at mean midday on the civil day 1875 February 7 and ends at mean midday on the civil day 1875 February 8.

The conversion from Washington Mean Astronomical Time to Universal Time is carried out in the following way :

<u>Example</u>

Given WMAT = 1875 February 7 10h 14m 23s

(1) Add 12h to get Washington Mean Civil Time

WMCT = 1875 February 7 22h 14m 23s

(2) Add the longitude of the observatory, expressed in hours, minutes and seconds and measured in a positive sense westwards.

The longitude of USNO is +5h 08m 15.71s

Thus the Universal Time is 1875 February 7   22h 14m 23s
+ 5h 08m 16s

= 1875 February 8   3h 22m 39s

So 1875 February 7 10h 14m 23s WMAT corresponds to 1875 February 8 3h 22m 39s Universal Time.

4.1.2 CONVERSION OF LST TO UNIVERSAL TIME

All observations published by H. Struve and G. Struve are measured using Local (Apparent) Sidereal Time. The conversion from LST to UT is illustrated by the following example, taken from G. Struve (Heft 2, page 20).

Example

Given Local (Apparent) Sidereal Time = 1916 January 11d 5h 10m 57s and λ (Babelsberg) = - 0h 52m 25.49s West

(1) This instant falls somewhere during the Astronomical Day of January 11 i.e. between about January 11d 18h UT and January 12d 6h UT.

(2) Strictly speaking, we should apply the correction for nutation (Equation of the Equinoxes) at the start of the calculation, but it is so small that it can be applied to the final derived UT. We shall neglect it in this example.

(3) Greenwich Sidereal Time = Local Sidereal Time + Longitude West

= 5h 10m 57s - 0h 52m 25s

= 4h 18m 32s

(4) Julian Day Number for 1916 January 11d 12h UT = 2420874.0

d = JD - 2415020.0 = 5854.0

T_u = d/36525 = 0.160273785

$$GMST_o = 23925^s.836 + 8640184^s.542 \ T_u + 0^s.0929 \ T_u^2$$

$$= 7^h \ 18^m \ 40^s.9$$

GMST at 1916 January 11d 12h UT = $19^h \ 18^m \ 40^s.9$

(5) Elapsed interval of Sidereal Time

$\Delta(ST) = (24^h +)\ 4^h\ 18^m\ 32^s - 19^h\ 18^m\ 40^s.9$

$= 8^h\ 59^m\ 51^s$

(6) Elapsed interval of Universal Time

$\Delta(UT) = \Delta(ST) \times \alpha \qquad \alpha = 0.99726\ 95664$

$= 8^h\ 58^m\ 23^s$

Thus the time is JD 2420874 + $8^h\ 58^m\ 23^s$ = 2420874.37388

4.1.3 CONVERSION OF UT TO LOCAL SIDEREAL TIME

We require the Local Apparent Sidereal Time at the instant of each observation in order to calculate the topocentric correction vector i.e. the topocentric position vector of the geocentre referred to the True Equator and Equinox of Date. For those observations where the LST is not explicitly given (that is, all data except that published by Struve father and son) we must calculate the Local Sidereal Time from the Universal Time of the observation. The procedure is described in the Explanatory Supplement and also in standard works on positional and observational astronomy such as Smart, so we do not provide details here.

4.1.4 UNIVERSAL TIME AND EPHEMERIS TIME

All of the observations of the satellites of Saturn are measured using time scales which are defined by the rotation of the Earth on its axis with respect to the Mean Equator and Equinox of date. The fundamental time scale so defined is called Universal Time.

The dynamical theories describing the motion of the satellites have a different time scale as their independent argument. This is Ephemeris Time and it is defined (and determined) by observations of the Sun, Moon and planets. That is to say, ET is defined by the orbital motions of several bodies in the Solar System, principally the Moon. Hence ET is defined by a different physical system to that defining UT. This difference is manifested by the behaviour of the quantity $\Delta T = ET - UT$. ΔT is non-zero and it is not constant : at the present time it has a value of about 60s and is increasing by approximately 1 second per year. A graph of ΔT together with a table of values from 1621 to 1972 is given in the Explanatory Supplement (pp 90 - 91).

In order to obtain a valid argument for our chosen dynamical theory, whether it is analytic or numeric, we must add ΔT to the UT of each observation so that we shall have a time expressed in the (dynamical) ET scale.

4.1.5 LIGHT-TIME AND THE TOPOCENTRIC POSITION OF SATURN

So far, all conversion and correction operations have been carried out on the topocentric time of the observation. We have the Ephemeris Time at which the observation was made on the Earth. However, we are observing a system which is some 8 or 9 AU distant and hence because the speed of light is finite, we see the satellite system as it was about 70 minutes ago. This time-lag is called light-time : it is the time taken for light from the system to reach the observer and it must be subtracted from the ET of the observation in order to yield the time argument with which we enter the dynamical theory.

The light-time is determined by an iterative process. It is assumed that we know the heliocentric position vector of the observer at the ET of the observation and that we possess a heliocentric theory of the motion of Saturn. The procedure may be represented as the following algorithm :

(1) Let the Ephemeris Time of the observation at the Earth be t and the topocentric position vector of the centre of the Sun at this instant be $\underline{R}$; as a first approximation set the light-time τ to zero : $\tau = 0$.

(2) Enter the heliocentric theory of Saturn with time argument $t - \tau$. Write the position vector (referred to the True Equator and Equinox of Date) as $\underline{r} = \underline{r}(t - \tau)$.

(3) Calculate the light-time from

$$[1] \qquad \tau = \beta | \underline{R} + \underline{r} |$$

where β is the light-time for unit distance, $5.77559 \; 10^{-3}$ days per AU.

(4) Repeat steps (2) and (3) until successive values of τ converge. This is the light-time to be subtracted from t in order to obtain the time argument for the dynamical models. The vector $\underline{R} + \underline{r}$ gives the topocentric position vector of Saturn for the observation.

4.1.6 ERROR IN THE TIME

A number of simplifying assumptions have been made when calculating the time argument for each observation. Since we are describing a dynamical system, an error in the time will cause a corresponding error in the Saturnicentric positions of the satellites and hence also in their positions as seen from the Earth. In this section we consider the size of such errors and their effect upon the reduction of the observations.

The errors are of two types :

(1) Systematic and/or random errors in the calculation of the time of the observation i.e. at the Earth before correction for light-time. Such errors may result from several sources.

- Error in the time given in the original reference.
- Error in the longitude adopted to convert LST and WMAT to UT.
- Error in the value adopted for ΔT.
- Error caused by neglecting the equation of the equinoxes ($\Delta\psi.\cos\varepsilon$) in the conversion of Local Apparent Sidereal Time to UT.
- Rounding error in the Julian Date ($0^{d}.00001 \simeq 0^{s}.86$).

The effect of random errors may be determined to first order by assuming that the satellite is moving in a circular orbit. We may calculate the distance moved by a satellite in its orbit in a given small time interval Δt. The distance is

[2] $\Delta x = a.n.\Delta t$

where a = semi-major axis of the orbit

n = mean motion of the satellite.

On the further assumption that the satellite system is viewed from an average distance of 8.5 AU then we can calculate the arc subtended by a distance Δx as seen from the Earth. This arc $\Delta\phi$ is the maximum error in the observed position of each satellite relative to Saturn due to an error Δt in the time argument of the theory. We may write

[3] $$\Delta\phi = \Delta x/8.5.$$

The arc $\Delta\phi$ corresponding to Δt = 1 second is given for each satellite in the table below. The error Δt required to produce a $\Delta\phi$ of 0.1 arc seconds is also given, expressed in days and in seconds.

The effect of an error in the time

Satellite	$\Delta\phi/''$ (for Δt=1s)	Δt (for $\Delta\phi=0''.1$)	
		days	seconds
Mimas	0.0023	0.000498	43.0
Enceladus	0.0021	0.000565	48.8
Tethys	0.0018	0.000628	54.3
Dione	0.0016	0.000711	61.5
Rhea	0.0014	0.000841	72.6
Titan	0.00090	0.00128	111
Hyperion	0.00082	0.00141	122
Iapetus	0.00053	0.00219	189

Clearly, for the outer satellites an error in the time of up to 10 seconds will not alter the observed position by more than $0''.01$. The observations are rather insensitive to small errors in the time.

(2) Error in the light time : we have assumed that the light time from the observer to each of the satellites is equal to that from the observer to the centre of Saturn. This is equivalent to the assumption that the satellites and Saturn are all equidistant from the observer. Evidently, depending upon the position in its orbit, each satellite will be closer or further than the centre of Saturn and so will require a slightly different light time. In a completely rigorous calculation we would use the iterative process described in the previous section upon each satellite in order to obtain individual light times, but we will show that the simplifying assumption does not introduce serious errors.

Consider the following diagram, which represents the orbit of a satellite (taken to be circular to first order).

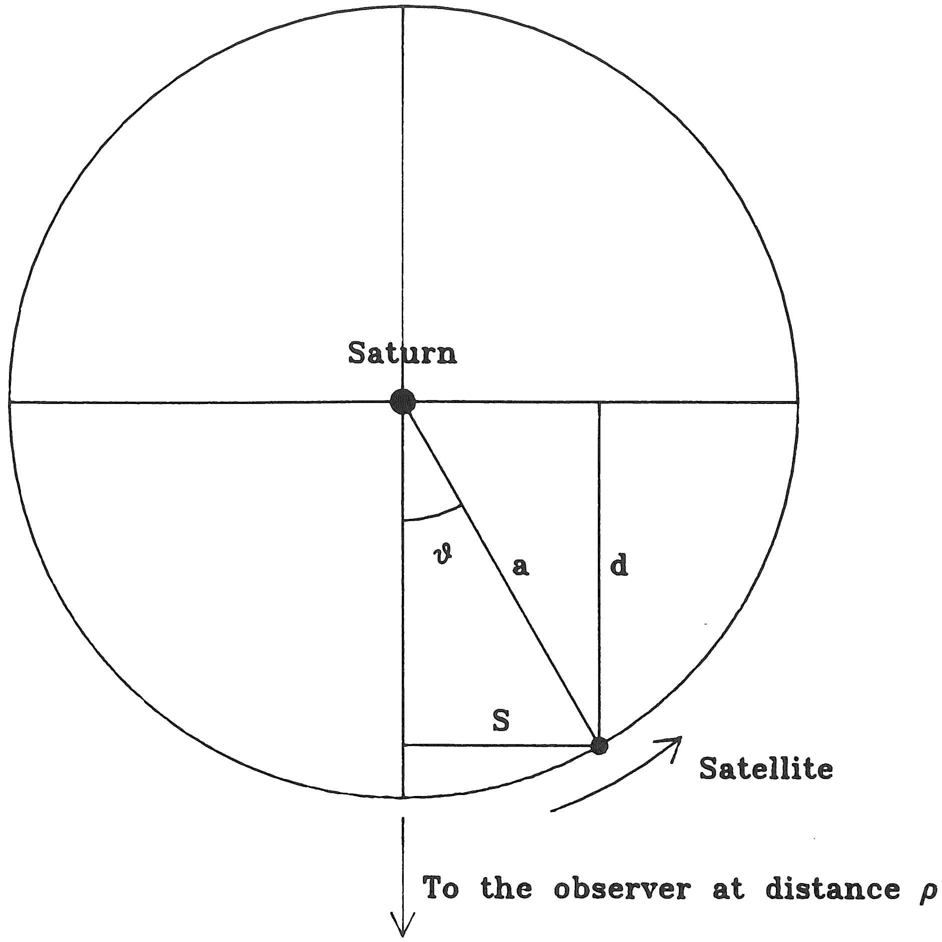

Figure 8. Light time error

$\Delta s = a.\cos\vartheta.\Delta\vartheta$ $\Delta\vartheta = n.\Delta t$

$\Delta t = -d/c$ $\Delta\psi = \Delta s/\rho$

Thus

[4] $\Delta\psi = -(na^2/\rho c)\cos^2\vartheta.$

The following table gives the value of $x = na^2/\rho c$ for each satellite. This is the maximum value of $\Delta\psi$. We assume $\rho = 8.5$ AU and $c = 173.14$ AU/day. This gives $x = na^2/1472 = 880.4\ a^2/P$ where P is the orbital period in days.

Light-time error

Satellite	x/arc-seconds
Mimas	0.0014
Enceladus	0.0016
Tethys	0.0018
Dione	0.0020
Rhea	0.0024
Titan	0.0037
Hyperion	0.0041
Iapetus	0.0063

We see that the effect of the error in the light-time is quite negligible for all the satellites.

4.2 COORDINATE SYSTEMS AND REFERENCE FRAMES

Each observation of the position of one satellite relative to another is made in a topocentric coordinate system which is unique and peculiar to the time and place of the observation. It is based upon the True Equator and Equinox of date but it includes the effects of aberration and of refraction insofar as they alter the relative positions of the two satellites. We may call this the 'O - frame'.

The dynamical theory describing the motions of the satellites is, by contrast, set in a fixed coordinate system (the 'I - frame') based upon <u>either</u> the Mean Equator and Equinox of the epoch B1950.0 <u>or</u> upon the equator plane of Saturn and its intersection with the Earth's Mean Equator of B1950.0. In order to compare the dynamical theory with observations it is necessary to apply a transformation to convert the position of the satellite in the I-frame into an observed position in the O-frame. The transformation consists of four stages :

(1) A rotation to convert from the Mean Equator and Equinox of B1950.0 (or Saturn's equator and the Earth's Mean Equator of B1950.0) to the True Equator and Equinox of Date.

After this stage we have Saturnicentric positions referred to the True Equator and Equinox of Date. The coordinate axes are now parallel to those used to express the topocentric position vector of Saturn.

(2) A translation to shift the origin from the centre of Saturn to the observer (i.e. the topocentre). This translation is carried out by adding the topocentric position vector of the centre of Saturn ($\underline{R}$) to the Sa-

turnicentric position vector of each satellite ($\underline{r}$) as shown in the following diagram.

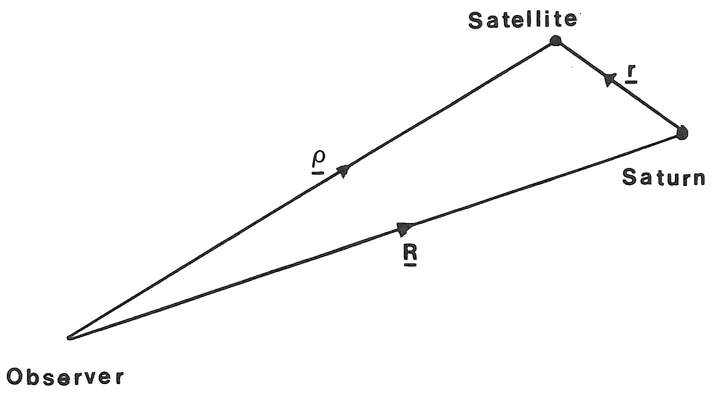

Figure 9. Topocentric position

We now have the topocentric position of each satellite referred to the True Equator and Equinox of date. However, it has been assumed that the satellite is at rest with respect to the observer. This is not so. Hence we require :

(3) A correction to allow for the velocity of the satellite relative to the observer. This correction is actually a Lorentz transformation in a simplified form and is called **aberration**.

(4) Finally, we must make allowance for the optical properties of the Earth's atmosphere through which, of course, all observation are made. The elevation of celestial bodies above the horizon is affected by atmospheric refraction. The size of the refraction depends upon the elevation itself.

Each of these stages are described fully in the following sections.

4.2.1 ROTATION TO THE TRUE EQUATOR AND EQUINOX OF DATE

We may assume that the Saturnicentric positions of the satellites given by the dynamical model are referred to the equator of Saturn and its intersection with the Earth's Mean Equator of B1950.0. This is the system used in the numeric integration program 'Titan' and it is defined thus :

1. The coordinate system has the centre of Saturn as its origin.

2. The equator plane of Saturn (i.e. its plane of axial symmetry) is the xy-plane of the coordinate system. This, together with (1), implies that the z-axis of the system is identical to Saturn's axis of rotational symmetry.

3. The direction of the x-axis of the system is defined by the ascending node of the equator plane of Saturn upon the Earth's Mean Equator of B1950.0

If we define

N = the Right Ascension of the ascending node of the equator plane of Saturn on the Earth's Mean Equator of B1950.0, referred to the Mean Equinox of B1950.0

I = the inclination of the equator plane of Saturn to the Earth's Mean Equator of B1950.0

then we may immediately write a transformation matrix to convert a vector from the reference frame defined above (the I-frame) to the Mean Equator and Equinox of B1950.0.

Let $(X, Y, Z)_I$ be the components of any vector referred to the integration frame and let $(X, Y, Z)_{B1950.0}$ be the components of the same vector referred to the Mean Equator and Equinox of B1950.0.

Then we may write

[5]
$$\begin{aligned} X_{1950.0} &= R_{11}X_I + R_{21}Y_I + R_{31}Z_I \\ Y_{1950.0} &= R_{12}X_I + R_{22}Y_I + R_{32}Z_I \\ Z_{1950.0} &= R_{13}X_I + R_{23}Y_I + R_{33}Z_I \end{aligned}$$

where the components of the matrix R are defined thus

$$[6] \quad \begin{aligned} R_{11} &= + \cos N &&= -0.6215608247 \\ R_{12} &= - \sin N \cos I &&= -0.7780537554 \\ R_{13} &= + \sin N \sin I &&= +0.0910741178 \end{aligned}$$

$$[7] \quad \begin{aligned} R_{21} &= + \sin N &&= +0.7780537554 \\ R_{22} &= + \cos N \cos I &&= -0.6173459053 \\ R_{23} &= - \cos N \sin I &&= +0.0722626596 \end{aligned}$$

$$[8] \quad \begin{aligned} R_{31} &= 0 &&= +0.0000000000 \\ R_{32} &= + \sin I &&= +0.1162599970 \\ R_{33} &= + \cos I &&= +0.9932188143 \end{aligned}$$

The numerical values of the coefficients are those used in the reduction of the observations and are based upon the following assumed values for I and N.

$$N = 8^h \; 33^m \; 43^s$$

$$I = 6^\circ \; 40' \; 35''$$

The second operation we must perform upon the position is the transformation from the Mean Equator and Equinox of B1950.0 to the True Equator and Equinox of Date. This transformation is a rotation and has two parts.

(1) Precession, which is the transformation from the Mean Equator and Equinox of B1950.0 to the Mean Equator and Equinox of Date.

(2) Nutation, which is the transformation from the Mean Equator and Equinox of Date to the True Equator and Equinox of Date.

Both transformations can be written in matrix form. The components of the precession matrix may be expressed as slowly-varying polynomials in time and we use the expressions given on page 34 of the Explanatory Supplement. We adopt a conventional matrix notation.

$$\pi_{11} = X_x \qquad \pi_{21} = X_y \qquad \pi_{31} = X_z$$

[6]

$$\pi_{12} = Y_x \qquad \pi_{22} = Y_y \qquad \pi_{32} = Y_z$$

$$\pi_{13} = Z_x \qquad \pi_{23} = Z_y \qquad \pi_{33} = Z_z$$

The components of the nutation matrix may be written in the form given on page 43 of the Explanatory Supplement or in the rigorous form given below.

$$\nu_{11} = + \cos\Delta\psi$$
$$\nu_{12} = - \cos\varepsilon_o . \sin\Delta\psi$$
$$\nu_{13} = - \sin\varepsilon_o . \sin\Delta\psi$$

$$\nu_{21} = + \cos\varepsilon . \sin\Delta\psi$$
$$\nu_{22} = + \cos\varepsilon . \cos\varepsilon_o . \cos\Delta\psi + \sin\varepsilon . \sin\varepsilon_o$$
$$\nu_{23} = + \cos\varepsilon . \sin\varepsilon_o . \cos\Delta\psi + \sin\varepsilon . \cos\varepsilon_o$$

$$\nu_{31} = + \sin\varepsilon . \sin\Delta\psi$$
$$\nu_{32} = + \sin\varepsilon . \cos\varepsilon_o . \cos\Delta\psi + \cos\varepsilon . \sin\varepsilon_o$$
$$\nu_{33} = + \sin\varepsilon . \sin\varepsilon_o . \cos\Delta\psi + \cos\varepsilon . \cos\varepsilon_o$$

where $\Delta\psi$ and $\Delta\varepsilon$ are respectively the nutation in longitude and in obliquity and are evaluated using the series given on pages 44-45 of the Explanatory Supplement.

ε_o = mean obliquity of date

$\varepsilon = \varepsilon_o + \Delta\varepsilon$ = true obliquity of date

We may combine the precession matrix and the nutation matrix for any given date to obtain the precession-nutation matrix Π defined by

$$[7] \qquad \Pi_{ij} = \nu_{i1}\pi_{1j} + \nu_{i2}\pi_{2j} + \nu_{i3}\pi_{3j}.$$

4.2.2 TRANSLATION OF THE ORIGIN TO THE OBSERVER

This is a very simple transformation : to obtain the topocentric coordinates of the satellite we add the Saturnicentric coordinates of that satellite to the topocentric coordinates of Saturn, which have been determined as part of the light time calculation. At this stage, all coordinates should be referred to the True Equator and Equinox of Date.

4.2.3 ABERRATION

The effect called aberration arises from the fact that the observer has a small but finite velocity relative to the observed object. Thus the observed direction to the object is not the same as its instantaneous geometrical direction but is given by the resultant of the velocity vector of the light from the object and the velocity of the observer relative to it.

A detailed account of aberration may be found in Smart, Brouwer and Clemence, Explanatory Supplement and other works on positional astronomy. It is sufficient to note here that aberration can change the position of objects near the ecliptic by up to 20 arc seconds. However, its effect varies only a little over a small area of the sky and so we need only consider the effect of differential aberration. A table for calculating differential aberration is given on pages 52-53 of the Explanatory Supplement. For objects near the ecliptic the following maximum values apply (Astronomical Almanac page B21).

$$\text{Change in } \Delta\alpha/0''.01 = -0.570 \cos(H+\alpha)\sec\delta.\Delta\alpha$$
$$-0.570 \sin(H+\alpha)\sec\delta\tan\delta.\Delta\delta$$

$$\text{Change in } \Delta\delta/0''.01 = +0.570 \sin(H+\alpha)\sin\delta.\Delta\alpha$$
$$-0.570 \cos(H+\alpha)\cos\delta.\Delta\delta$$

where $\Delta\alpha$ and $\Delta\delta$ are in arc-minutes.

(Note that the coefficient 0.570 becomes 0.00950 when $\Delta\alpha$ and $\Delta\delta$ are in arc-seconds.)

For $\delta = +23^\circ$ we have

$$\Delta\alpha/0''.01 = -0.62 \cos(H+\alpha).\Delta\alpha$$
$$-0.26 \sin(H+\alpha).\Delta\delta$$

$$\Delta\delta/0''.01 = +0.22 \sin(H+\alpha).\Delta\alpha - 0.52 \cos(H+\alpha).\Delta\delta.$$

For Iapetus the separation can be approximately 9'.5 giving $\Delta\alpha \simeq$ 10'.3/1.414 and $\Delta\delta \simeq$ 9'.5/1.414 so that the maximum effect is of the order of 0".06 for the outermost satellite .

We can regard the effect of differential aberration as proportional to the separation and so it is most important for the outer satellites. A change of 0".06 in the relative positions of two satellites should not be neglected in the reduction of the observations.

We calculate the effect of aberration upon the positions of the satellites using the formulae adapted from pages 156-160 of the Explanatory Supplement.

Writing $\Delta\alpha$ = correction to the Right Ascension and $\Delta\delta$ = correction to the Declination then

$$[8] \quad \begin{aligned} \Delta\alpha &= h.\sin(H+\alpha_o).\sec\delta_o \\ \Delta\delta &= h.\cos(H+\alpha_o).\sin\delta_o + i.\cos\delta_o \end{aligned}$$

where
$$h.\sin H = C$$
$$h.\cos H = D$$
$$i = C.\tan\varepsilon = 0.43382.C \text{ for practical purposes.}$$

C and D are the aberration Day Numbers formed from the velocity of the Earth relative to Saturn. The heliocentric velocity of the Earth is calculated using the subroutine BARVEL (Stumpff 1980) whilst the velocity of Saturn is calculated by a 9-point Lagrange differentiation formula operating upon a set of heliocentric coordinates tabulated at equal intervals. Denoting by (x', y', z') the components of the velocity of the Earth relative to Saturn, referred to the Mean Equator and Equinox of Date, then the Day Numbers are given by :

$$C = +y'/c \qquad [9]$$
$$D = -x'/c$$

where $1/c = 5.7756.10^{-3}$ days/AU.

As in the calculation of the Day Numbers in the Astronomical Almanac, the motions of the Earth and of Saturn are assumed to lie entirely within the plane of the ecliptic. Saturn's orbit is inclined at about 2°.5 to the ecliptic and this introduces an error of approximately (V/c)tan2°.5 into the aberration in the Declination, where V is the orbital speed of Saturn. The relative size of this error compared to the total effect of aberration is (V/U)tan2°.5 where U is the orbital speed of the Earth. This amounts to 0.014 or about 1% of the total aberration. The greatest error that this will produce in the differential aberration for any satellite pair is thus 0".06 x 0.014 = 0".0008 which is entirely negligible.

4.2.4 REFRACTION

The variation of the refractive index of the Earth's atmosphere causes the light rays from astronomical objects to be deflected during the passage through the atmosphere. A thorough account of refraction can be found in Smart (1978) and we adopt the formula given by Smart, namely

[10] $$R = R(\zeta) = z - \zeta = 58''.294 \tan\zeta \quad - \quad 0''.0668 \tan^3\zeta$$

where z = true zenith distance (i.e. unaffected by refraction)

ζ = observed zenith distance.

Thus the effect of refraction is to increase the zenith distance of an object by an amount R. This causes the observed Right Ascension and Declination of the object to be changed by a small amount $\Delta\alpha$, $\Delta\delta$.

As in the case of aberration, we are concerned with differential effects and thus with the change in refraction over a small area of the sky. Consider two objects close together but with different zenith distances. Using the notation above we may write

$$z_1 = \zeta_1 + R(\zeta_1)$$
$$z_2 = \zeta_2 + R(\zeta_2).$$

Thus

[11] $$z_2 - z_1 = \zeta_2 - \zeta_1 + R(\zeta_2) - R(\zeta_1)$$

which we may write to first order as

[12] $$z_2 - z_1 = \zeta_2 - \zeta_1 + (\zeta_2 - \zeta_1)\frac{dR}{d\zeta}.$$

$$= (\zeta_2 - \zeta_1)(1 + \frac{dR}{d\zeta})$$

where $dR/d\zeta$ is evaluated at $\zeta = \zeta_1$.

Writing $R = A.\tan\zeta + B.\tan^3\zeta$ then we have

[13] $$\frac{dR}{d\zeta} = A + (A + 3B)\tan^2\zeta + 3B\tan^4\zeta$$

and the coefficients A and B are, in radians, $2.8262.10^{-4}$ and $-3.24.10^{-7}$ respectively.

The difference in the zenith distance between the two objects is affected by refraction in the following way.

$$(z_2 - z_1) - (\zeta_2 - \zeta_1)$$

$$= (\zeta_2 - \zeta_1)(A + (A + 3B)\tan^2\zeta + 3B\tan^4\zeta)$$

To second order we have

[14] $$z_2 - z_1 = (\zeta_2 - \zeta_1)(1 + \frac{dR}{d\zeta}) + \frac{1}{2}(\zeta_2 - \zeta_1)^2 \frac{d^2R}{d\zeta^2}$$

and

[15] $$\frac{d^2R}{d\zeta^2} = 2(A + 3B)\tan\zeta + (2A + 18B)\tan^3\zeta + 12B\tan^5\zeta.$$

As an example of the magnitude of the effect of differential refraction we may take the extreme case of two objects only 30° above the horizon and differing in elevation by 600 arc seconds.

Thus $\zeta_1 = 60°$ and $\zeta_2 - \zeta_1 = 2.9\times10^{-3}$ radians

$$dR/d\zeta = 1.12\times10^{-3}$$

$$d^2R/d\zeta^2 = 3.82\times10^{-3}.$$

Hence the first-order term is 3.2×10^{-6} radians = 0".67 while the second-order term is 1.6×10^{-8} = 0".003. The first-order effect of refraction is to decrease the difference in zenith distance by about one part in a thousand. Clearly, the second-order term is quite negligible.

In practise we wish to calculate the effect of refraction upon the observed RA and Dec of the objects, and hence upon their relative positions.

We begin by writing the formulae given in the Explanatory Supplement (page 26) which relate the elevation and azimuth of an object to its Declination and Local Hour Angle (h = Local Sidereal Time - R.A.).

$$\begin{aligned} \cos\delta\sin h &= -\cos a\cos A &&= \ell \\ \cos\delta\cos h &= \sin a\cos\phi - \cos a\cos A\sin\phi &&= m \qquad [19] \\ \sin\delta &= \sin a\sin\phi + \cos a\cos A\cos\phi &&= n \end{aligned}$$

$$\cos a \sin A = -\cos\delta \sin h$$

[20] $$\cos a \cos A = \sin\delta \cos\phi - \cos\delta \cos h \sin\phi$$

$$\sin a = \sin\delta \sin\phi + \cos\delta \cos h \cos\phi$$

We take the partial derivatives of the first two equations with respect to the elevation a

[21]
$$-\sin\delta \sin h \frac{\partial\delta}{\partial a} + \cos\delta \cos h \frac{\partial h}{\partial a} = \frac{\partial \ell}{\partial a}$$

$$-\sin\delta \cos h \frac{\partial\delta}{\partial a} - \cos\delta \sin h \frac{\partial h}{\partial a} = \frac{\partial m}{\partial a}$$

which may be solved to yield

$$\sin\delta \frac{\partial\delta}{\partial a} = -\sin h \frac{\partial \ell}{\partial a} - \cos h \frac{\partial m}{\partial a}$$

$$\cos\delta \frac{\partial h}{\partial a} = -\sin h \frac{\partial m}{\partial a} + \cos h \frac{\partial \ell}{\partial a}.$$

It is preferable to employ $\partial n/\partial a$ in order to evaluate $\partial\delta/\partial a$. Thus

[22] $$\cos\delta \frac{\partial\delta}{\partial a} = \frac{\partial n}{\partial a}$$

because the factor $\sin\delta$ causes problems for objects near to the celestial equator.

We also re-write $\partial h/\partial a$ as $-\partial\alpha/\partial a$ and we finally obtain

$$\cos^2\delta \; \frac{\partial\alpha}{\partial a} = - \cos\phi \sin A$$

[20]

$$\cos\delta \; \frac{\partial\delta}{\partial a} = \sin\phi \cos a - \cos\phi \sin a \cos A.$$

Using these derivatives we may calculate the change in α and δ due to a small change (Δa) in the elevation :

$$\Delta\alpha = \frac{\partial\alpha}{\partial a} \Delta a = \frac{\partial\alpha}{\partial a} \times R$$

[21]

$$\Delta\delta = \frac{\partial\delta}{\partial a} \Delta a = \frac{\partial\delta}{\partial a} \times R.$$

These formulae represent first-order corrections. Rigorous corrections may be obtained by calculating the pre-refraction elevation and azimuth of each object, adding the refraction to the elevation and then re-calculating the hour-angle and declination. However, first-order corrections are adequate for all practical purposes.

4.2.5 POSITION ANGLE AND SEPARATION

Most of the observations of the satellites of Saturn made in the period 1874 to 1947 are in the form of position angle (P) and separation (s) of one satellite relative to another. We must relate such P and s measures to the topocentric RA and Dec of a pair of satellites.

Consider the topocentric spherical triangle whose vertices are defined by the two satellites (denoted A and B) and the North Celestial Pole. We require the position angle of B, the observed satellite, with respect to A, the reference satellite.

In Figure 10 on page 107 the arc PA is the co-declination of A, that is $90^\circ - \delta_A$, and PB is the co-declination of B.

The angle APB is the difference in the Right Ascensions of the two satellites : $\alpha_B - \alpha_A$. Note that the order of the satellites is important in this angle. The difference must be formed in the sense observed minus reference.

The arc AB is the separation of the two satellites and the angle PAB is the position angle of B with respect to A. Position angle is measured from North towards East, that is in an anti-clockwise sense as indicated in the diagram.

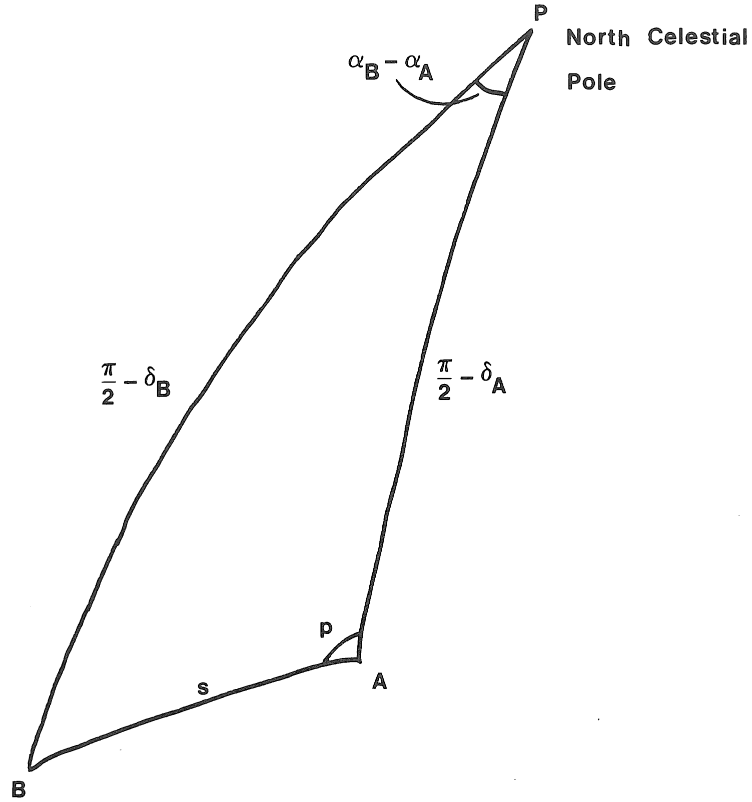

Figure 10. Position Angle and Separation

Using the formulae of spherical trigonometry (Explanatory Supplement page 472) we may write

$$\lambda = \sin s \sin p \quad = \cos\delta_B \sin(\alpha_B - \alpha_A)$$

[22] $$\mu = \sin s \cos p \quad = \sin\delta_B \cos\delta_A - \cos\delta_B \sin\delta_A \cos(\alpha_B - \alpha_A)$$

$$\nu = \cos s \quad = \sin\delta_B \sin\delta_A + \cos\delta_B \cos\delta_A \cos(\alpha_B - \alpha_A).$$

Since sin s is always positive, we may deduce the quadrant of p by taking the sign of λ as the sign of sin p and the sign of μ as the sign of cos p. The following table gives the appropriate quadrant for p.

λ	μ	p	Quadrant
+	+	$0° \le p \le 90°$	1^{st}
+	-	$90° \le p \le 180°$	2^{nd}
-	-	$180° \le p \le 270°$	3^{rd}
-	+	$270° \le p \le 360°$	4^{th}

4.2.6 FIRST-ORDER CORRECTION TO P AND S FOR ABERRATION AND REFRACTION

If we neglect the effects of aberration and refraction upon the topocentric Right Ascension and Declination of the satellites then the deduced position angle and separation will be in error by a small amount, typically up to 0".7 for the outermost satellite Iapetus. Analysis of the available data has shown that the correction to be made to p and s for the combined effects of aberration and refraction generally does not exceed 0.2% of the datum itself even for objects observed at elevations less than 40°. For this reason we may regard aberration and refraction as small corrections to position angle and separation due to small changes in the R.A. and Dec of the satellites. If we suppose that aberration and refraction combine to alter the topocentric R.A. and Dec of the satellites by amounts $\Delta\alpha_A$, $\Delta\delta_A$, $\Delta\alpha_B$, $\Delta\delta_B$ then we may write

$$\Delta p = \frac{\partial p}{\partial \alpha_A}\Delta\alpha_A + \frac{\partial p}{\partial \delta_A}\Delta\delta_A + \frac{\partial p}{\partial \alpha_B}\Delta\alpha_B + \frac{\partial p}{\partial \delta_B}\Delta\delta_B$$

[23]

$$\Delta s = \frac{\partial s}{\partial \alpha_A}\Delta\alpha_A + \frac{\partial s}{\partial \delta_A}\Delta\delta_A + \frac{\partial s}{\partial \alpha_B}\Delta\alpha_B + \frac{\partial s}{\partial \delta_B}\Delta\delta_B.$$

In order to determine Δp and Δs it is evident that we need the partial derivatives of p and s with respect to the R.A. and Dec of the two satellites. These derivatives are also required in the differential correction process which is used to calculate improved values of the parameters of the dynamical model, and expressions for these partial derivatives are given in the following sections.

4.3 COMPARISON OF OBSERVATION WITH THEORY : DIFFERENTIAL CORRECTION

Comparison of a dynamical theory with observational data serves two purposes.

1. It allows us to evaluate how well the theory represents the dynamics of the real system which it is intended to model.

2. It enables us to improve the theory so that it is a closer model of the real system. This improvement may take the form of additional terms in an analytic theory, but more often it involves making small corrections to the numerical values of the parameters upon which the theory is based (i.e. orbital elements, or starting conditions of a

numerical integration). This method of improving a dynamical theory can be put into practise using the technique of differential corrections to the parameters.

Suppose we have a theory which provides the positions of a number of satellites at any given time t. The theory contains a set of parameters (orbital elements) $e_1, e_2, \ldots, e_N$ and so in a mathematical sense the coordinates of each satellite are functions of those parameters and also of the time argument. Any observable quantity, say a position angle p, is a function of the coordinates of the satellites and hence it is also a function of the parameters of the theory via the coordinates.

With a chosen set of parameters we may calculate the value of the observed quantity at some instant using the dynamical theory. This is called the computed ('C') value of p and it is denoted as p_c. At the same instant we have a value of p which has been obtained by observing the real satellite system. This is the observed ('O') value and is denoted as p_o. The difference between the two, in the sense observed-minus-computed, is called the O-C or residual and is denoted as Δp :

[24] $$\Delta p = p_o - p_c$$

We assume that this difference is due mainly to errors in the adopted values of the parameters and we seek to make corrections to the parameters $\Delta e_1, \Delta e_2, \ldots, \Delta e_N$ where

[25] $$\Delta e_i = (e_i)_o - (e_i)_c$$

is the difference between the 'correct' or 'best' value and the value adopted in the theory. Thus we may write

$$[26] \qquad \Delta p = \frac{\partial p}{\partial e_1} \Delta e_1 + \frac{\partial p}{\partial e_2} \Delta e_2 + \dots + \frac{\partial p}{\partial e_N} \Delta e_N \quad .$$

In this equation, the left-hand side is a known quantity, as are the derivatives $\partial p/\partial e_1$, $\partial p/\partial e_2$ etc. The corrections Δe_1, Δe_2, ..., Δe_N are not known : they are the quantities which we wish to determine.

Such an equation is known as an equation of condition and it is the basis of differential correction theory. Each observation yields an equation of condition. In this context, a position angle measurement and a separation measurement are regarded as separate observations and each can be used to form an equation of condition.

When we have a large number of observations, we can form many equations of condition and so we have a number of simultaneous linear equations whose unknowns are the correction Δe_i. In most cases, we choose to combine the equations of condition into a set of normal equations in order to obtain a least-squares solution.

In the following sections we derive expressions for the partial derivatives of observed quantities with respect to the Saturnicentric coordinates of the satellites produced by the dynamical models. Such derivatives may be used with any theory which gives rectangular coordinates of the satellites in a fixed Saturnicentric reference frame.

4.4 PARTIAL DERIVATIVES OF POSITION ANGLE AND SEPARATION

In this section we derive the expressions for the partial derivatives of position angle and separation with respect to the Saturnicentric coordinates of the two observed objects. These coordinates are referred to the True Equator and Equinox of Date, but both they and the partial derivatives may be readily be converted to the fixed reference frame of the theory or the integration.

Consider the spherical triangle (on the topocentric sky sphere) formed by the North Celestial Pole and the two satellites. Refer to the diagram, which shows the triangle as seen by the observer from the inside of the sphere. Let the topocentric spherical coordinates of the objects be ρ_A, α_A, δ_A and ρ_B, α_B, δ_B and let p,s be the position angle and separation of B relative to A. Position angle on the celestial sphere is measured from the north, eastwards. Then we have

$$\begin{aligned} \lambda &= \sin s \sin p &&= \cos\delta_B \sin(\alpha_B - \alpha_A) \\ \mu &= \sin s \cos p &&= \sin\delta_B \cos\delta_A - \cos\delta_B \sin\delta_A \cos(\alpha_B - \alpha_A) \qquad [27] \\ \nu &= \cos s &&= \sin\delta_B \sin\delta_A + \cos\delta_B \cos\delta_A \cos(\alpha_B - \alpha_A). \end{aligned}$$

By differentiating λ and μ with respect to any parameter ω and rearranging, we obtain

$$[28] \quad \begin{aligned} \sin s \frac{\partial p}{\partial \omega} &= \cos p \frac{\partial \lambda}{\partial \omega} - \sin p \frac{\partial \mu}{\partial \omega} \\ \cos s \frac{\partial s}{\partial \omega} &= \sin p \frac{\partial \lambda}{\partial \omega} + \cos p \frac{\partial \mu}{\partial \omega} \end{aligned}$$

and it is evident that

$$[29] \quad \begin{aligned} \frac{\partial p}{\partial \lambda} &= \frac{\cos p}{\sin s} & \frac{\partial p}{\partial \mu} &= - \frac{\sin p}{\sin s} \\ \frac{\partial s}{\partial \lambda} &= \frac{\sin p}{\cos s} & \frac{\partial s}{\partial \mu} &= \frac{\cos p}{\cos s}. \end{aligned}$$

We form the derivatives of λ and μ with respect to the R.A. and Dec of the two objects thus :

$$[30] \quad \begin{aligned} \frac{\partial \lambda}{\partial \alpha_A} &= - \frac{\partial \lambda}{\partial \alpha_B} = - \cos \delta_B \cos (\alpha_B - \alpha_A) \\ \frac{\partial \lambda}{\partial \delta_A} &= 0 \\ \frac{\partial \lambda}{\partial \delta_B} &= - \sin \delta_B \sin (\alpha_B - \alpha_A) \end{aligned}$$

$$[31] \quad \begin{aligned} \frac{\partial \mu}{\partial \alpha_A} &= - \frac{\partial \mu}{\partial \alpha_B} = - \lambda \sin \delta_A \\ \frac{\partial \mu}{\partial \delta_A} &= - \nu \\ \frac{\partial \mu}{\partial \delta_B} &= \cos \delta_B \cos \delta_A + \sin \delta_B \sin \delta_A \cos (\alpha_B - \alpha_A) \end{aligned}$$

and we may now calculate the derivatives of p and s with respect to α_A, δ_A, α_B, δ_B.

[35] $$\frac{\partial p}{\partial \alpha_A} = \frac{\partial p}{\partial \lambda}\frac{\partial \lambda}{\partial \alpha_A} + \frac{\partial p}{\partial \lambda}\frac{\partial \mu}{\partial \alpha_A}$$

and likewise for the other derivatives.

We may seek the derivatives of the R.A. and Dec with respect to the Saturnicentric coordinates of the satellites. For each satellite we have

[36] $$\begin{aligned} \rho \cos \delta \cos \alpha &= x + X \\ \rho \cos \delta \sin \alpha &= y + Y \\ \rho \sin \delta &= z + Z \end{aligned}$$

where (x,y,z) are the Saturnicentric coordinates of the satellite and (X,Y,Z) are the topocentric coordinates of Saturn, all referred to the True Equator and Equinox of Date.

We obtain the following derivatives :

[37] $$\begin{aligned} \frac{\partial \alpha}{\partial x} &= - \frac{\sin \alpha}{\rho \cos \delta} \\ \frac{\partial \alpha}{\partial y} &= + \frac{\cos \alpha}{\rho \cos \delta} \\ \frac{\partial \alpha}{\partial z} &= 0 \end{aligned}$$

$$\frac{\partial\delta}{\partial x} = - \frac{\sin\delta\cos\alpha}{\rho}$$

[38] $$\frac{\partial\delta}{\partial y} = - \frac{\sin\delta\sin\alpha}{\rho}$$

$$\frac{\partial\delta}{\partial z} = \frac{\cos\delta}{\rho}$$

We may form these derivatives for both satellites and then we combine them with derivatives such as $\partial p/\partial\alpha$ to obtain

[39] $$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial\alpha}\frac{\partial\alpha}{\partial x} + \frac{\partial p}{\partial\delta}\frac{\partial\delta}{\partial x}$$

and so forth. Note that α_A and δ_A are independent of x_B, y_B, z_B, and α_B and δ_B are independent of x_A, y_A, z_A. Hence we do not include terms such as $\partial\alpha_A/\partial x_B$ since they are zero.

At this point we have the partial derivatives of position angle and separation with respect to the Cartesian coordinates of the satellites in a system where all quantities are referred to the True Equator and Equinox of Date. The dynamical model of the satellite system is referred to a fixed coordinate system and so we must transform the derivatives to that coordinate system. We recall that the coordinates in the reference frame of the dynamical theory may be related to the True Equator and Equinox of Date via a transformation matrix which incorporates the instantaneous effect of precession and nutation and (in the case of the numeric integration) a constant rotation transformation. If we denote this matrix by M then we may write

$$[37] \qquad \begin{aligned} x &= M_{11}X + M_{12}Y + M_{13}Z \\ y &= M_{21}X + M_{22}Y + M_{23}Z \\ z &= M_{31}X + M_{32}Y + M_{33}Z \end{aligned}$$

where (x,y,z) are the components of a vector referred to the True Equator and Equinox of Date and (X,Y,Z) are the components of that vector in the reference frame of the dynamical model.

Now for any observed quantity p, we have

$$[38] \qquad \frac{\partial p}{\partial X} = \frac{\partial p}{\partial x}\frac{\partial x}{\partial X} + \frac{\partial p}{\partial y}\frac{\partial y}{\partial X} + \frac{\partial p}{\partial z}\frac{\partial z}{\partial X}$$

and likewise for $\partial p/\partial Y$ and $\partial p/\partial Z$.

The derivatives $\partial x/\partial X$, $\partial y/\partial X$, $\partial z/\partial X$ etc. may be recognised as the components of the matrix M. For example :

$$[39] \qquad \begin{aligned} \partial x/\partial X &= M_{11} \\ \partial y/\partial X &= M_{21} \\ \partial z/\partial X &= M_{31}. \end{aligned}$$

We may now calculate the partial derivatives of position angle and separation with respect to the Saturnicentric coordinates produced by the dynamical model. These are the derivatives that were sought in this section.